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ON THE GENERAL SOLUTIONS OF TRANSVERSELY ISOTROPIC ELASTICITY†

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Abstract—In this paper we give the generalized Boussinesq–Galerkin general solution of transversely isotropic elasticity, as well as its simplified forms in two special cases. And we prove the completeness of the Lekhnitskii–Hu–Nowacki solution and the Elliott–Lodge solution in such cases that s_0^2, s_1^2 and s_2^2 are possibly equal to each other. © 1998 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

Due to the anisotropy of composite materials, the study of anisotropic elasticity is becoming gradually important along with the wide use of composite materials. Transversely isotropic material is a noticeable kind of anisotropic material. Several studies on transversely isotropic elasticity have appeared so far, such as Lekhnitskii (1940, 1981), Hu (1953), Nowacki (1954), Lodge (1955), Elliott (1948), Alexsandrov and Soloviev (1978), Ding and Xu (1988), Horgan and Simmonds (1991) and Fabrikant (1996).

It is the purpose of this paper to continue our previous work Wang and Wang (1995). In Section 2, for transversely isotropic elasticity we will give a new kind of general solution, which is similar in form to the Boussinesq–Galerkin general solution of isotropic elasticity and which will be called generalized B–G solution. In Section 3, we will derive the simplified forms of generalized B–G solutions in two special cases, in which the first one corresponds to the condition $\delta = 0$ in Wang *et al.* (1994), and the solution in the second case can degenerate into the B–G solution of isotropic elasticity. In Sections 4 and 5, we will extend the theorems 5–7 in Wang and Wang (1995) to the cases that s_0^2, s_1^2 and s_2^2 are possibly equal to each other and give the proof of the completeness of the corresponding Lekhnitskii–Hu–Nowacki solution (LHN) and Elliott–Lodge solution (E–L).

2. GENERALIZED B–G GENERAL SOLUTION

In a rectangular coordinate system (x, y, z) the generalized Hooke's law of a transversely isotropic body is

$$\begin{cases} \sigma_x = A_{11}\dfrac{\partial u}{\partial x} + A_{12}\dfrac{\partial v}{\partial y} + A_{13}\dfrac{\partial w}{\partial z}, \\ \sigma_y = A_{12}\dfrac{\partial u}{\partial x} + A_{11}\dfrac{\partial v}{\partial y} + A_{13}\dfrac{\partial w}{\partial z}, \\ \sigma_z = A_{13}\dfrac{\partial u}{\partial x} + A_{13}\dfrac{\partial v}{\partial y} + A_{33}\dfrac{\partial w}{\partial z}, \end{cases} \quad \begin{cases} \tau_{yz} = A_{44}\left(\dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z}\right), \\ \tau_{zx} = A_{44}\left(\dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}\right), \\ \tau_{xy} = A_{66}\left(\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y}\right), \end{cases} \tag{1}$$

where σ_x, $\sigma_y, \ldots, \tau_{xy}$ are stress components, u, v, w the components of displacement, A_{11}, $A_{12}, \ldots, A_{66}$ elastic constants and

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$$2A_{66} = A_{11} - A_{12}. \tag{2}$$

The substitution of eqn (1) into equilibrium equations without body force yields the operator equation in terms of the displacements.

$$\mathbf{Pu} = \mathbf{0}, \tag{3}$$

where $\mathbf{u} = (u, v, w)^{\mathrm{T}}$ (the superscript T denotes the transpose) and $\mathbf{P}$ is a 3×3 differential operator matrix

$$\mathbf{P} = \begin{bmatrix} \Lambda + \alpha_1\,\partial_x^2 + \alpha_2\,\partial_z^2 & \alpha_1\,\partial_x\,\partial_y & \alpha_3\,\partial_x\,\partial_z \\ \alpha_1\,\partial_x\,\partial_y & \Lambda + \alpha_1\,\partial_y^2 + \alpha_2\,\partial_z^2 & \alpha_3\,\partial_y\,\partial_z \\ \alpha_3\,\partial_x\,\partial_z & \alpha_3\,\partial_y\,\partial_z & \alpha_2\Lambda + \alpha_4\,\partial_z^2 \end{bmatrix}, \tag{4}$$

in which

$$\partial_x = \frac{\partial}{\partial x}, \quad \partial_y = \frac{\partial}{\partial y}, \quad \partial_z = \frac{\partial}{\partial z}, \quad \Lambda = \partial_x^2 + \partial_y^2, \tag{5}$$

$$\alpha_1 = \frac{A_{12} + A_{66}}{A_{66}}, \quad \alpha_2 = \frac{A_{44}}{A_{66}}, \quad \alpha_3 = \frac{A_{13} + A_{44}}{A_{66}}, \quad \alpha_4 = \frac{A_{33}}{A_{66}}. \tag{6}$$

Assume that $\mathbf{Q} = [Q_{ij}]$ is the "adjoint matrix" of $\mathbf{P}$ and its components are

$$\begin{aligned} Q_{11} &= (\Lambda + \alpha_1\,\partial_y^2 + \alpha_2\,\partial_z^2)(\alpha_2\Lambda + \alpha_4\,\partial_z^2) - \alpha_3^2\,\partial_y^2\,\partial_z^2, \\ Q_{22} &= (\Lambda + \alpha_1\,\partial_x^2 + \alpha_2\,\partial_z^2)(\alpha_2\Lambda + \alpha_4\,\partial_z^2) - \alpha_3^2\,\partial_x^2\,\partial_z^2, \\ Q_{12} = Q_{21} &= -\alpha_1\alpha_2\nabla_a^2\,\partial_x\,\partial_y, \\ Q_{13} = Q_{31} &= -\alpha_3\nabla_0^2\,\partial_x\,\partial_z, \\ Q_{23} = Q_{32} &= -\alpha_3\nabla_0^2\,\partial_y\,\partial_z, \\ Q_{33} &= (1 + \alpha_1)\nabla_0^2(\Lambda + \beta\,\partial_z^2), \end{aligned} \tag{7a–f}$$

where $\beta = (\alpha_2/1 + \alpha_1)$,

$$\nabla_0^2 = \Lambda + \frac{1}{s_0^2}\partial_z^2, \quad \frac{1}{s_0^2} = \alpha_2, \tag{8}$$

$$\nabla_a^2 = \Lambda + a\,\partial_z^2, \quad a = \frac{\alpha_1\alpha_4 - \alpha_3^2}{\alpha_1\alpha_2}. \tag{9}$$

Consider the second-degree equation of $1/s^2$

$$(1 + \alpha_1)\alpha_2\frac{1}{s^4} - (\alpha_2^2 + \alpha_4 + \alpha_1\alpha_4 - \alpha_3^2)\frac{1}{s^2} + \alpha_2\alpha_4 = 0, \tag{10}$$

whose two roots $1/s_1^2$ and $1/s_2^2$ are not negative reals, which has been proved by Lekhnitskii (1981).

In Wang and Wang (1995) it is proved that for eqn (3), there is the following complete solution

$$\mathbf{u} = \mathbf{Q}\mathbf{\Phi}, \tag{11}$$

in which $\mathbf{\Phi} = (\varphi_1, \varphi_2, \varphi_3)^{\mathrm{T}}$ and satisfies the equation

$$\nabla_0^2\nabla_1^2\nabla_2^2\mathbf{\Phi} = \mathbf{0}, \quad \nabla_i^2 = \Lambda + \frac{1}{s_i^2}\partial_z^2, \quad i = 0, 1, 2. \tag{12a,b}$$

Now we will rewrite the form of solution (11). Set

$$\begin{aligned} K_{11} &= Q_{11} + \alpha_1\alpha_2\nabla_a^2\,\partial_x^2, \\ K_{22} &= Q_{22} + \alpha_1\alpha_2\nabla_a^2\,\partial_y^2. \end{aligned} \tag{13}$$

Inserting (7a,b) into (13) and using eqns (10) and (12b), we can get

$$K_{11} = K_{22} = (1+\alpha_1)\alpha_2\nabla_1^2\nabla_2^2. \tag{14}$$

From (13) and (14) the first two expressions of solution (11) are changed into

$$\begin{aligned} u &= (1+\alpha_1)\alpha_2\nabla_1^2\nabla_2^2\varphi_1 - \alpha_1\alpha_2\frac{\partial}{\partial x}\left[\nabla_a^2\left(\frac{\partial\varphi_1}{\partial x} + \frac{\partial\varphi_2}{\partial y}\right) + \frac{\alpha_3}{\alpha_1\alpha_2}\nabla_0^2\frac{\partial\varphi_3}{\partial z}\right], \\ v &= (1+\alpha_1)\alpha_2\nabla_1^2\nabla_2^2\varphi_2 - \alpha_1\alpha_2\frac{\partial}{\partial y}\left[\nabla_a^2\left(\frac{\partial\varphi_1}{\partial x} + \frac{\partial\varphi_2}{\partial y}\right) + \frac{\alpha_3}{\alpha_1\alpha_2}\nabla_0^2\frac{\partial\varphi_3}{\partial z}\right], \end{aligned} \tag{15}$$

while the third one of solution (11) is

$$w = (1+\alpha_1)\nabla_0^2\nabla_b^2\varphi_3 - \alpha_3\frac{\partial}{\partial z}\left[\nabla_0^2\left(\frac{\partial\varphi_1}{\partial x} + \frac{\partial\varphi_2}{\partial y}\right) + \frac{\alpha_3}{\alpha_1\alpha_2}\nabla_0^2\frac{\partial\varphi_3}{\partial z}\right], \tag{16}$$

where

$$\nabla_b^2 = \Lambda + b\,\partial_z^2, \quad b = \frac{\alpha_1\alpha_2^2 + \alpha_3^2}{(1+\alpha_1)\alpha_1\alpha_2}. \tag{17}$$

Let

$$G_1 = (1+\alpha_1)\alpha_2\varphi_1, \quad G_2 = (1+\alpha_1)\alpha_2\varphi_2, \quad G_3 = (1-\alpha_1)\varphi_3. \tag{18}$$

Solutions (15) and (16) become

$$\begin{aligned} u &= \nabla_1^2\nabla_2^2 G_1 - \frac{\alpha_1}{1+\alpha_1}\frac{\partial}{\partial x}\left[\nabla_a^2\left(\frac{\partial G_1}{\partial x} + \frac{\partial G_2}{\partial y}\right) + \frac{\alpha_3}{\alpha_1}\nabla_0^2\frac{\partial G_3}{\partial z}\right], \\ v &= \nabla_1^2\nabla_2^2 G_2 - \frac{\alpha_1}{1+\alpha_1}\frac{\partial}{\partial y}\left[\nabla_a^2\left(\frac{\partial G_1}{\partial x} + \frac{\partial G_2}{\partial y}\right) + \frac{\alpha_3}{\alpha_1}\nabla_0^2\frac{\partial G_3}{\partial z}\right], \\ w &= \nabla_0^2\nabla_b^2 G_3 - \frac{\alpha_3}{(1+\alpha_1)\alpha_2}\frac{\partial}{\partial z}\left[\nabla_0^2\left(\frac{\partial G_1}{\partial x} + \frac{\partial G_2}{\partial y}\right) + \frac{\alpha_3}{\alpha_1}\nabla_0^2\frac{\partial G_3}{\partial z}\right], \end{aligned} \tag{19}$$

in which $\mathbf{G} = (G_1, G_2, G_3)^{\mathrm{T}}$ and satisfies the equation

$$\nabla_0^2\nabla_1^2\nabla_2^2\mathbf{G} = \mathbf{0}. \tag{20}$$

Thus, for transversely isotropic elasticity, there are the complete solutions (19) and (20), which are called generalized B–G solutions and very similar to the B–G solution of isotropic elasticity [see formula (45) of this paper].

If setting

$$F = -\frac{\alpha_3}{1+\alpha_1}\nabla_0^2 G_3, \tag{21}$$

eqn (19) may be rewritten as

$$\begin{aligned} u &= \nabla_1^2\nabla_2^2 G_1 - \frac{\alpha_1}{1+\alpha_1}\frac{\partial}{\partial x}\nabla_a^2\left(\frac{\partial G_1}{\partial x}+\frac{\partial G_2}{\partial y}\right)+\frac{\partial^2 F}{\partial x\,\partial y}, \\ v &= \nabla_1^2\nabla_2^2 G_2 - \frac{\alpha_1}{1+\alpha_1}\frac{\partial}{\partial y}\nabla_a^2\left(\frac{\partial G_1}{\partial x}+\frac{\partial G_2}{\partial y}\right)+\frac{\partial^2 F}{\partial y\,\partial z}, \\ w &= -\frac{\alpha_3}{(1+\alpha_1)\alpha_2}\frac{\partial}{\partial z}\nabla_0^2\left(\frac{\partial G_1}{\partial x}+\frac{\partial G_2}{\partial y}\right)-\alpha\left(\Lambda+\beta\frac{\partial^2}{\partial z^2}\right)F, \end{aligned} \tag{22}$$

where

$$\nabla_1^2\nabla_2^2 F = 0, \quad \alpha = \frac{1+\alpha_1}{\alpha_3}, \quad \beta = \frac{\alpha_2}{1+\alpha_1}. \tag{23}$$

3. TWO SPECIAL FORMS OF GENERALIZED B–G GENERAL SOLUTION

Next we will deduce the simplified forms of generalized B–G general solution, respectively, when $s_0^2 = s_1^2$ and $s_0^2 = s_1^2 = s_2^2$.

Case 1 : $s_0^2 = s_1^2$

Now $(1/s_0^2) = \alpha_2$ is one root of eqn (10). Inserting α_2 into the above-mentioned equation, we derive an expression of the relation among α_i $(i = 1, 2, 3, 4)$

$$\alpha_3^2+\alpha_1\alpha_2^2-\alpha_1\alpha_4 = 0. \tag{24}$$

Using the condition (24) in eqn (9), we can get

$$a = \alpha_2, \quad \nabla_a^2 = \nabla_0^2 = \nabla_1^2. \tag{25a,b}$$

Moreover, considering the relation between roots and coefficients of eqn (10) together with eqns (9) and (17), we find

$$a+b = \frac{1}{s_1^2}+\frac{1}{s_2^2}. \tag{26}$$

From (25a), it turns out that

$$b = \frac{1}{s_2^2} = \frac{\alpha_4}{(1+\alpha_1)\alpha_2}, \quad \nabla_b^2 = \nabla_2^2. \tag{27}$$

In eqns (19) and (20), setting $\mathbf{B} = \nabla_0^2(\mathbf{i}G_1 + \mathbf{j}G_2 + \mathbf{k}G_3)$ and inserting (25) and (27) into them, we can obtain the following B–G type solution

$$\mathbf{u} = \nabla_b^2\mathbf{B} - \frac{\alpha_1}{1+\alpha_1}\nabla_\gamma(\nabla_\delta \cdot \mathbf{B}),$$
$$\nabla_0^2\nabla_b^2\mathbf{B} = \mathbf{0}, \tag{28}$$

where

$$\nabla_\gamma = \mathbf{i}\,\partial_x + \mathbf{j}\,\partial_y + \mathbf{k}\frac{\alpha_3}{\alpha_1\alpha_2}\partial_z,$$
$$\nabla_\delta = \mathbf{i}\,\partial_x + \mathbf{j}\,\partial_y + \mathbf{k}\frac{\alpha_3}{\alpha_1}\partial_z. \tag{29}$$

When $s_0^2 = s_2^2$, solutions (28) and (29) are also tenable.

If (6) is substituted into the condition (24), then it is observed that

$$(A_{13}+A_{44})^2A_{66} + (A_{12}+A_{66})A_{44}^2 - (A_{12}+A_{66})A_{33}A_{66} = 0. \tag{30}$$

The condition (30) and the solution (28) are exactly the condition (2.19) and the solution (3.1) in Wang *et al.* (1994), respectively.

As a result, if the condition (24) is satisfied, eqn (3) can be changed into

$$\nabla_0^2\mathbf{u} + \alpha_1\nabla_\gamma(\nabla_\delta \cdot \mathbf{u}) = \mathbf{0}. \tag{31}$$

Using the same method as in Wang *et al.* (1994), from eqn (31) we can also get the B–G type solutions (28) and (29), as well as the following Papkovich–Neuber type solution

$$\mathbf{u} = \mathbf{P} - \frac{\alpha_1}{2(1+\alpha_1)}\nabla_\gamma(P_0 + \mathbf{r}_* \cdot \mathbf{P}),$$
$$\nabla_2^2\mathbf{P} = \mathbf{0},$$
$$\nabla_1^2 P_0 = -\mathbf{r}_* \cdot \nabla_1^2\mathbf{P}, \quad \mathbf{r}_* = \mathbf{i}x + \mathbf{j}y + \mathbf{k}\frac{\alpha_3}{b\alpha_1}z. \tag{32}$$

Case 2: $s_0^2 = s_1^2 = s_2^2$

In this case both a and b are equal to α_2. Thus, from (9) and (17) it follows that

$$\alpha_3 = \pm\alpha_1\alpha_2, \quad \alpha_4 = (1+\alpha_1)\alpha_2^2. \tag{33a,b}$$

And then from (33b) we have

$$\alpha_2 = \pm\sqrt{\frac{\alpha_4}{1+\alpha_1}}. \tag{34}$$

Since α_2 is one root of eqn (10), the roots of which are not negative reals, the negative symbol in (34) should be omitted. Considering $1+\alpha_1 = (A_{11}/A_{66}) > 0$, we get

$$\alpha_2(1+\alpha_1) = \sqrt{\alpha_4(1+\alpha_1)}. \tag{35}$$

In addition (33a) can be rewritten as

$$\alpha_3 \pm \alpha_2 = \pm\alpha_2(1+\alpha_1). \tag{36}$$

The insertion of (35) into (36) yields

$$\alpha_3 \pm \alpha_2 = \pm\sqrt{\alpha_4(1+\alpha_1)}. \tag{37}$$

Again substituting (6) into (37), we get

$$A_{13}+2A_{44} = \sqrt{A_{11}A_{33}}, \quad A_{13} = -\sqrt{A_{11}A_{33}}. \tag{38a,b}$$

Due to the positive definite of strain energy, $A_{11}A_{33}-A_{13}^2 > 0$. Therefore (38b) is incorrect, i.e., the negative root in (33a) should be omitted, which implies

$$\alpha_3 = \alpha_1\alpha_2. \tag{39}$$

Thus, the general solutions (28) and (29) are changed into

$$\begin{aligned} \mathbf{u} &= \nabla_0^2\mathbf{B} - \frac{\alpha_1}{1+\alpha_1}\nabla(\nabla_\beta \cdot \mathbf{B}), \\ \nabla_0^2\nabla_0^2\mathbf{B} &= \mathbf{0}, \end{aligned} \tag{40}$$

in which

$$\begin{aligned} \nabla &= \mathbf{i}\,\partial_x + \mathbf{j}\,\partial_y + \mathbf{k}\,\partial_z, \\ \nabla_\beta &= \mathbf{i}\,\partial_x + \mathbf{j}\,\partial_y + \mathbf{k}\alpha_2\,\partial_z. \end{aligned} \tag{41}$$

Let

$$\begin{aligned} \hat{\mathbf{u}} &= \mathbf{i}u + \mathbf{j}v + \mathbf{k}\sqrt{\alpha_2}w, \\ \hat{\mathbf{B}} &= \mathbf{i}B_1 + \mathbf{j}B_2 + \mathbf{k}\sqrt{\alpha_2}B_3, \\ \nabla_0 &= \mathbf{i}\,\partial_x + \mathbf{j}\,\partial_y + \mathbf{k}\sqrt{\alpha_2}\,\partial_z. \end{aligned} \tag{42}$$

Then (40) becomes

$$\begin{aligned} \hat{\mathbf{u}} &= \nabla_0^2\hat{\mathbf{B}} - \frac{\alpha_1}{1+\alpha_1}\nabla_0(\nabla_0 \cdot \hat{\mathbf{B}}), \\ \nabla_0^2\nabla_0^2\hat{\mathbf{B}} &= \mathbf{0}. \end{aligned} \tag{43}$$

When the material is isotropic,

$$\alpha_2 = 1, \quad \frac{\alpha_1}{1+\alpha_1} = \frac{1}{2(1-\nu)}. \tag{44}$$

Hence, eqn (43) changes into

$$\mathbf{u} = \nabla^2\mathbf{B} - \frac{1}{2(1-\nu)}\nabla(\nabla\cdot\mathbf{B}),$$
$$\nabla^2\nabla^2\mathbf{B} = \mathbf{0}. \tag{45}$$

This is the B–G solution of isotropic elasticity.

4. COMPLETENESS OF THE LHN SOLUTION

First, it is easy to verify that G_1, G_2 and G_3 in the generalized B–G solutions (19) and (20) can be substituted by the following $\hat{G}_1$, $\hat{G}_2$ and $\hat{G}_3$, respectively

$$\hat{G}_1 = G_1 + \alpha_3\frac{\partial^2 h_3}{\partial x\,\partial z},$$
$$\hat{G}_2 = G_2 + \alpha_3\frac{\partial^2 h_3}{\partial y\,\partial z},$$
$$\hat{G}_3 = G_3 + \left(\Lambda + \frac{\alpha_4}{\alpha_2}\partial_z^2\right)h_3, \tag{46a,b,c}$$

where

$$\nabla_0^2\nabla_1^2\nabla_2^2 h_3 = 0. \tag{47}$$

Using the above-mentioned nonuniqueness of G_1, G_2 and G_3, Wang and Wang (1995) has proved that if the elastic region is z-convex and s_0^2, s_1^2 and s_2^2 are not equal to each other, the LHN solution

$$u = \frac{\partial^2 F}{\partial x\,\partial z} - \frac{\partial\phi_0}{\partial y},$$
$$v = \frac{\partial^2 F}{\partial y\,\partial z} + \frac{\partial\phi_0}{\partial x},$$
$$w = -\alpha(\Lambda + \beta\,\partial_z^2)F, \quad \alpha = \frac{1+\alpha_1}{\alpha_3}, \quad \beta = \frac{\alpha_2}{1+\alpha_1},$$
$$\nabla_1^2\nabla_2^2 F = 0, \quad \nabla_0^2\phi_0 = 0, \tag{48}$$

is complete. In this section we will further prove that if s_0^2, s_1^2 and s_2^2, or two of which, are equal, the LHN solution is also complete. For this purpose we begin with proof of the following lemmas.

Lemma 1.

Assume the elastic region Ω is z-convex and $s_0^2 = s_1^2 = s_2^2$. Then there exists h_3 such that

$$\hat{G}_i = \hat{G}_i^{(0)} + z\hat{G}_i^{(1)} + z^2\hat{G}_i^{(2)}, \quad i = 1, 2, \tag{49}$$

in which

$$\frac{\partial \hat{G}_1^{(j)}}{\partial x}+\frac{\partial \hat{G}_2^{(j)}}{\partial y}=0, \quad j=0,1,2, \tag{50}$$

$$\nabla_0^2 \hat{G}_i^{(j)}=0, \quad i=1,2; \quad j=0,1,2. \tag{51}$$

Proof. According to Lemma 4 of the Appendix, G_i ($i=1,2$) can be expressed as

$$\begin{aligned} &G_i=G_i^{(0)}+zG_i^{(1)}+z^2G_i^{(2)}, \quad i=1,2, \\ &\nabla_0^2 G_i^{(j)}=0, \quad i=1,2; \quad j=0,1,2. \end{aligned} \tag{52a,b}$$

Similarly, we can suppose

$$\begin{aligned} &h_3=h^{(0)}+zh^{(1)}+z^2h^{(2)}, \\ &\nabla_0^2 h^{(j)}=0, \quad j=0,1,2. \end{aligned} \tag{53a,b}$$

The insertion of (52a) and (53a) into (46a,b) yields

$$\hat{G}_i=\hat{G}_i^{(0)}+z\hat{G}_i^{(1)}+z^2\hat{G}_i^{(2)}, \quad i=1,2, \tag{54}$$

where

$$\begin{aligned} &\hat{G}_i^{(0)}=G_i^{(0)}+\alpha_3\frac{\partial}{\partial x_i}\left(h^{(1)}+\frac{\partial h^{(0)}}{\partial z}\right), \\ &\hat{G}_i^{(1)}=G_i^{(1)}+\alpha_3\frac{\partial}{\partial x_i}\left(2h^{(2)}+\frac{\partial h^{(1)}}{\partial z}\right), \quad i=1,2, \\ &\hat{G}_i^{(2)}=G_i^{(2)}+\alpha_3\frac{\partial}{\partial x_i}\frac{\partial h^{(2)}}{\partial z}, \end{aligned} \tag{55}$$

in which $(\partial/\partial x_1)=(\partial/\partial x)$, $(\partial/\partial x_2)=(\partial/\partial y)$.

From eqns (55), (52b) and (53b) it follows that

$$\nabla_0^2 \hat{G}_i^{(j)}=0, \quad i=1,2; \quad j=0,1,2. \tag{56}$$

Again inserting (55) into (50), we obtain

$$\frac{\partial \hat{G}_1^{(2)}}{\partial x}+\frac{\partial \hat{G}_2^{(2)}}{\partial y}=\frac{\partial G_1^{(2)}}{\partial x}+\frac{\partial G_2^{(2)}}{\partial y}+\alpha_3\Lambda\frac{\partial h^{(2)}}{\partial z}=0, \tag{57}$$

$$\frac{\partial \hat{G}_1^{(1)}}{\partial x}+\frac{\partial \hat{G}_2^{(1)}}{\partial y}=\frac{\partial G_1^{(1)}}{\partial x}+\frac{\partial G_2^{(1)}}{\partial y}+\alpha_3\Lambda\left(2h^{(2)}+\frac{\partial h^{(1)}}{\partial z}\right)=0, \tag{58}$$

$$\frac{\partial \hat{G}_1^{(0)}}{\partial x}+\frac{\partial \hat{G}_2^{(0)}}{\partial y}=\frac{\partial G_1^{(0)}}{\partial x}+\frac{\partial G_2^{(0)}}{\partial y}+\alpha_3\Lambda\left(h^{(1)}+\frac{\partial h^{(0)}}{\partial z}\right)=0. \tag{59}$$

From Lemma 2 in the Appendix, there

exists $h^{(2)}$, which makes (57) reached and $\nabla_0^2 h^{(2)}=0$,

exists $h^{(1)}$, which makes (58) reached and $\nabla_0^2 h^{(1)}=0$,

$$\text{exists } h^{(0)}, \text{which makes (59) reached and } \nabla_0^2 h^{(0)} = 0. \tag{60}$$

So the lemma is proved.

Lemma 2.

If the elastic region Ω is z-convex, and two of s_0^2 s_1^2 and s_2^2 are equal, we might as well assume $s_0^2 = s_1^2 \neq s_2^2$, then there exists h_3 such that

$$\hat{G}_i = \hat{G}_i^{(0)} + z\hat{G}_i^{(1)} + \hat{G}_i^{(2)}, \quad i = 1,2, \tag{61}$$

where

$$\begin{aligned} &\frac{\partial \hat{G}_1^{(j)}}{\partial x} + \frac{\partial \hat{G}_2^{(j)}}{\partial y} = 0, \quad j = 0,1,2, \\ &\nabla_0^2 \hat{G}_i^{(j)} = 0, \quad j = 0,1; \quad i = 1,2, \\ &\nabla_2^2 \hat{G}_i^{(2)} = 0, \quad i = 1,2. \end{aligned} \tag{62}$$

From Lemma 5 of the Appendix we can set

$$\begin{aligned} &h_3 = h^{(0)} + zh^{(1)} + h^{(2)}, \\ &\nabla_0^2 h^{(j)} = 0, \quad j = 0,1; \quad \nabla_2^2 h^{(2)} = 0. \end{aligned} \tag{63}$$

In the similar way of the proof of Lemma 1, there exist $h^{(0)}$, $h^{(1)}$ and $h^{(2)}$ such that (61) and (62) are reached. The proof is omitted here.

Next, we come to prove the completeness of the LHN solution.

Theorem 1.

Assume the elastic region Ω is z-convex. Then the LHN solution (48) is complete.

Proof. It is shown previously that in eqns (19) and (20), G_1, G_2, G_3 can be replaced by $\hat{G}_1$, $\hat{G}_2$, $\hat{G}_3$ in eqn (46), respectively, i.e.

$$\begin{aligned} u &= \nabla_1^2\nabla_2^2\hat{G}_1 - \frac{\alpha_1}{1+\alpha_1}\frac{\partial}{\partial x}\left[\nabla_a^2\left(\frac{\partial \hat{G}_1}{\partial x} + \frac{\partial \hat{G}_2}{\partial y}\right) + \frac{\alpha_3}{\alpha_1}\nabla_0^2\frac{\partial \hat{G}_3}{\partial z}\right], \\ v &= \nabla_1^2\nabla_2^2\hat{G}_2 - \frac{\alpha_1}{1+\alpha_1}\frac{\partial}{\partial y}\left[\nabla_a^2\left(\frac{\partial \hat{G}_1}{\partial x} + \frac{\partial \hat{G}_2}{\partial y}\right) + \frac{\alpha_3}{\alpha_1}\nabla_0^2\frac{\partial \hat{G}_3}{\partial z}\right], \\ w &= \nabla_0^2\nabla_b^2\hat{G}_3 - \frac{\alpha_3}{(1+\alpha_1)\alpha_2}\frac{\partial}{\partial z}\left[\nabla_0^2\left(\frac{\partial \hat{G}_1}{\partial x} + \frac{\partial \hat{G}_2}{\partial y}\right) + \frac{\alpha_3}{\alpha_1}\nabla_0^2\frac{\partial \hat{G}_3}{\partial z}\right], \end{aligned} \tag{64}$$

in which $\hat{G}_i$ $(i = 1,2)$ are given by (49) or (61), $\hat{G}_3$ by (46c).

Let

$$\begin{aligned} A^{(j)} &= \int_{\mathbf{x}_0}^{\mathbf{x}} \hat{G}_2^{(j)}\,\mathrm{d}x - \hat{G}_1^{(j)}\,\mathrm{d}y + B^{(j)}\,\mathrm{d}z, \\ B^{(j)} &= \int_{\mathbf{x}_0}^{\mathbf{x}} \frac{\partial \hat{G}_2^{(j)}}{\partial z}\mathrm{d}x - \frac{\partial \hat{G}_1^{(j)}}{\partial z}\mathrm{d}y + s_j^2\left(-\frac{\partial \hat{G}_2^{(j)}}{\partial x} + \frac{\partial \hat{G}_1^{(j)}}{\partial y}\right)\mathrm{d}z, \end{aligned} \tag{65}$$

where $\mathbf{x}_0$ is some point of the region Ω and $\mathbf{x}$ is any point of the region Ω. Because of conditions (50) and (51) or (62), the linear integral of eqn (65) is independent of routes. In view of (65), we have

$$\hat{G}_1^{(j)} = -\frac{\partial A^{(j)}}{\partial y}, \quad \hat{G}_2^{(j)} = \frac{\partial A^{(j)}}{\partial x}, \quad j = 0, 1, 2, \tag{66}$$

$$\nabla_j^2 A^{(j)} = 0, \quad j = 0, 1, 2. \tag{67}$$

Set

$$\begin{aligned} A &= A^{(0)} + zA^{(1)} + z^2 A^{(2)}, \quad \text{when } s_0^2 = s_1^2 = s_2^2, \\ A &= A^{(0)} + zA^{(1)} + A^{(2)}, \quad \text{when } s_0^2 = s_1^2 \neq s_2^2. \end{aligned} \tag{68}$$

Equations (67) and (68) lead to

$$\nabla_0^2 \nabla_1^2 \nabla_2^2 A = 0. \tag{69}$$

While from eqns (49) or (61), (66) and (68) we get

$$\hat{G}_1 = -\frac{\partial A}{\partial y}, \quad \hat{G}_2 = \frac{\partial A}{\partial x}. \tag{70}$$

Applying (70) and writing $-[\alpha_3/(1+\alpha_1)]\nabla_0^2 \hat{G}_3$ as F in eqn (64), and then letting $\phi_0 = \nabla_1^2 \nabla_2^2 A$, we arrive at the LHN solution (48).

Moreover, it can be verified that solution (48) satisfies eqn (3). Therefore, the LHN solution is complete. Theorem 1 is proved.

When the material is isotropic, the introduction of (39) and (44) into (48) yields

$$\begin{aligned} u &= \frac{\partial^2 F}{\partial x \, \partial z} - \frac{\partial \phi_0}{\partial y}, \\ v &= \frac{\partial^2 F}{\partial y \, \partial z} + \frac{\partial \phi_0}{\partial x}, \\ w &= \frac{\partial^2 F}{\partial z^2} - 2(1-\nu)\nabla^2 F, \\ &\nabla^4 F = 0, \quad \nabla^2 \phi_0 = 0. \end{aligned} \tag{71}$$

5. COMPLETENESS OF THE E–L SOLUTION

When s_1^2 and s_2^2 are not equal to each other, Wang and Wang (1995) has proved the completeness of the following E–L solution with the aid of the LHN solution (48)

$$\begin{aligned} u &= \frac{\partial}{\partial x}(\phi_1 + \phi_2) - \frac{\partial \phi_0}{\partial y}, \\ v &= \frac{\partial}{\partial y}(\phi_1 + \phi_2) + \frac{\partial \phi_0}{\partial x}, \\ w &= \frac{\partial}{\partial z}(k_1 \phi_1 + k_2 \phi_2), \end{aligned} \tag{72}$$

where

$$k_i = -\frac{\alpha_2}{\alpha_3}+\frac{1+\alpha_1}{\alpha_3}\frac{1}{s_i^2}, \quad i = 1,2, \qquad \nabla_i^2\phi_i = 0, \quad i = 0,1,2, \tag{73}$$

and

$$k_1 k_2 = 1. \tag{74}$$

Next we will take the case of $s_1^2 = s_2^2$ into account.

Theorem 2.

Assume the elastic region Ω is z-convex and $s_1^2 = s_2^2$. Then the following solution is complete

$$u = \frac{\partial}{\partial x}(\phi_1+z\phi_2)-\frac{\partial\phi_0}{\partial y},$$

$$v = \frac{\partial}{\partial y}(\phi_1+z\phi_2)+\frac{\partial\phi_0}{\partial x},$$

$$w = \frac{\partial}{\partial z}(\phi_1+z\phi_2)-2\left(1+\frac{\alpha_2}{\alpha_3}\right)\phi_2,$$

$$\nabla_1^2\phi_i = 0, \quad i = 1,2; \quad \nabla_0^2\phi_0 = 0. \tag{75}$$

Proof. It is not difficult to verify that (75) satisfies eqn (3). So it is omitted. Now we prove that any solution of eqn (3) can be expressed as (75).

First, when $s_1^2 = s_2^2$, the solution of eqn (3) can be rewritten as (48). And from Lemma 3 of the Appendix, we can resolve F in (48) in the form

$$F = f_1+zf_2, \quad \nabla_1^2 f_i = 0, \quad i = 1,2. \tag{76a,b}$$

Set

$$\phi_1 = f_2+\frac{\partial f_1}{\partial z}, \quad \phi_2 = \frac{\partial f_2}{\partial z}. \tag{77}$$

Then from (77) and (76b), there is

$$\nabla_1^2\phi_i = 0, \quad i = 1,2. \tag{78}$$

After inserting (76a) into (48) and then using (77), we obtain the solution (75). So Theorem 2 is proved.

When the material is isotropic, (75) changes into

$$u = \frac{\partial}{\partial x}(\phi_1+z\phi_2)-\frac{\partial\phi_0}{\partial y},$$

$$v = \frac{\partial}{\partial y}(\phi_1+z\phi_2)+\frac{\partial\phi_0}{\partial x},$$

$$w = \frac{\partial}{\partial z}(\phi_1+z\phi_2)-4(1-\nu)\phi_2,$$

$$\nabla^2\phi_i = 0, \quad i = 0,1,2. \tag{79}$$

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APPENDIX

If every line segment parallel to the z-axis lies entirely in Ω whenever its end points belong to Ω, we call such a region Ω z-convex. In this Appendix we always assume that Ω is z-convex.

First we take the Lemma 2 in the Appendix of Wang and Wang (1995) as Lemma 1 herein

Lemma 1.

If the region Ω is z-convex and f satisfies.

$$\nabla_s^2 f = 0, \quad \text{in } \Omega, \tag{A1}$$

where s is not pure imaginary and

$$\nabla_s^2 = \Lambda + \frac{1}{s^2}\partial_z^2, \tag{A2}$$

there exists A such that

$$\begin{cases} \nabla_s^2 A = 0, \\ \dfrac{\partial^k A}{\partial z^k} = f, \end{cases} \quad \text{in } \Omega, \tag{A3}$$

where k are positive integers.

Applying Lemma 1, we can prove the following several lemmas:

Lemma 2.

In the same conditions as in Lemma 1, there exists B such that

$$\begin{cases} \nabla_s^2 B = 0, \\ \Lambda \dfrac{\partial B}{\partial z} = -f, \end{cases} \quad \text{in } \Omega. \tag{A4}$$

The proof is omitted.

Lemma 3.

If A satisfies

$$\nabla_s^2 \nabla_s^2 A = 0, \qquad \text{in } \Omega, \tag{A5}$$

there exist $A^{(0)}$ and $A^{(1)}$ such that

$$\begin{cases} A = A^{(0)} + zA^{(1)}, \\ \nabla_s^2 A^{(j)} = 0, \quad j = 0, 1, \end{cases} \quad \text{in } \Omega. \tag{A6}$$

Proof. Let

$$B = \nabla_s^2 A. \tag{A7}$$

Then

$$\nabla_s^2 B = 0. \tag{A8}$$

According to Lemma 1, there exists $A^{(1)}$ which satisfies

$$\begin{cases} \nabla_s^2 A^{(1)} = 0, \\ \dfrac{\partial A^{(1)}}{\partial z} = \dfrac{s^2}{2} B, \end{cases} \quad \text{in } \Omega. \tag{A9a,b}$$

From (A7) and (A9) we have

$$\nabla_s^2 A = \frac{2}{s^2} \frac{\partial A^{(1)}}{\partial z} = \nabla_s^2 (zA^{(1)}). \tag{A10}$$

Therefore,

$$\nabla_s^2 (A - zA^{(1)}) = 0. \tag{A11}$$

Setting

$$A^{(0)} = A - zA^{(1)}, \tag{A12}$$

then (A11) becomes

$$\nabla_s^2 A^{(0)} = 0. \tag{A13}$$

Equations (A12), (A13) and (A9a) together are the same as (A6).
The lemma is proved.

Lemma 4.
Assume

$$\nabla_s^2 \nabla_s^2 \nabla_s^2 A = 0, \quad \text{in } \Omega. \tag{A14}$$

Then there exist $A^{(0)}$, $A^{(1)}$ and $A^{(2)}$ such that

$$\begin{cases} A = A^{(0)} + zA^{(1)} + z^2 A^{(2)}, \\ \nabla_s^2 A^{(j)} = 0, \quad j = 0, 1, 2, \end{cases} \quad \text{in } \Omega. \tag{A15}$$

Proof. Letting

$$B = \nabla_s^2 A, \tag{A16}$$

then

$$\nabla_s^2 \nabla_s^2 B = 0. \tag{A17}$$

According to Lemma 3, there exist $B^{(0)}$ and $B^{(1)}$ which satisfy

$$B = B^{(0)} + zB^{(1)}, \quad \nabla_s^2 B^{(j)} = 0, \quad j = 0, 1. \tag{A18a,b}$$

From Lemma 1 there exist $A^{(2)}$, $A^{(1)}$, respectively, such that

$$\frac{4}{s^2}\frac{\partial A^{(2)}}{\partial z} = B^{(1)}, \quad \nabla_s^2 A^{(2)} = 0, \tag{A19a,b}$$

$$\frac{2}{s^2}\frac{\partial A^{(1)}}{\partial z} = B^{(0)} - \frac{2}{s^2}A^{(2)}, \quad \nabla_s^2 A^{(1)} = 0. \tag{A20a,b}$$

Based on (A19) and (A20), we get

$$\nabla_s^2(z^2 A^{(2)}) = z^2\nabla_s^2 A^{(2)} + \frac{1}{s^2}\left(2A^{(2)} + 4z\frac{\partial A^{(2)}}{\partial z}\right) = \frac{2}{s^2}A^{(2)} + zB^{(1)},$$

$$\nabla_s^2(zA^{(1)}) = z\nabla_s^2 A^{(1)} + \frac{2}{s^2}\frac{\partial A^{(1)}}{\partial z} = B^{(0)} - \frac{2}{s^2}A^{(2)}. \tag{A21}$$

By adding the two expressions of (A21) and using (A16) and (A18a)

$$\nabla_s^2(zA^{(1)} + z^2 A^{(2)}) = B^{(0)} + zB^{(1)} = B = \nabla_s^2 A. \tag{A22}$$

Setting

$$A^{(0)} = A - zA^{(1)} - z^2 A^{(2)} \tag{A23}$$

and considering (A22) and (A23), we have

$$\nabla_s^2 A^{(0)} = 0. \tag{A24}$$

Equations (A23), (A24), (A20b) and (A19b) together are exactly (A15).
Thus, the lemma is proved.
Similarly, we have:

Lemma 5.
If

$$\nabla_0^2\nabla_0^2\nabla_2^2 A = 0, \quad \text{in } \Omega, \tag{A25}$$

then there exist $A^{(0)}$, $A^{(1)}$, $A^{(2)}$ such that

$$\begin{cases} A = A^{(0)} + zA^{(1)} + A^{(2)}, \\ \nabla_0^2 A^{(j)} = 0, \quad j = 0, 1; \quad \nabla_0^2 A^{(2)} = 0, \end{cases} \quad \text{in } \Omega. \tag{A26}$$

Proof. Let

$$B = \nabla_2^2 A. \tag{A27}$$

Then

$$\nabla_0^2\nabla_0^2 B = 0. \tag{A28}$$

According to Lemma 3, there exist $B^{(0)}$ and $B^{(1)}$ which satisfy

$$B = B^{(0)} + zB^{(1)}, \quad \nabla_0^2 B^{(j)} = 0, \quad j = 0, 1. \tag{A29a,b}$$

From Lemma 1 there exist $A^{(1)}$ and $A^{(0)}$, respectively, such that

$$\left(\frac{1}{s_2^2} - \frac{1}{s_0^2}\right)\frac{\partial^2 A^{(1)}}{\partial z^2} = B^{(1)}, \quad \nabla_0^2 A^{(1)} = 0, \tag{A30a,b}$$

$$\left(\frac{1}{s_2^2} - \frac{1}{s_0^2}\right)\frac{\partial^2 A^{(0)}}{\partial z^2} = B^{(0)} - \frac{2}{s_2^2}\frac{\partial A^{(1)}}{\partial z}, \quad \nabla_0^2 A^{(0)} = 0. \tag{A31a,b}$$

Thus, we have

$$\nabla_2^2 A^{(0)} = \nabla_0^2 A^{(0)} + \left(\frac{1}{s_2^2} - \frac{1}{s_0^2}\right)\frac{\partial^2 A^{(0)}}{\partial z^2} = B^{(0)} - \frac{2}{s_2^2}\frac{\partial A^{(1)}}{\partial z},$$

$$\nabla_2^2(zA^{(1)}) = z\nabla_2^2 A^{(1)} + \frac{2}{s_2^2}\frac{\partial A^{(1)}}{\partial z}$$

$$= z\nabla_0^2 A^{(1)} + z\left(\frac{1}{s_2^2} - \frac{1}{s_0^2}\right)\frac{\partial^2 A^{(1)}}{\partial z^2} + \frac{2}{s_2^2}\frac{\partial A^{(1)}}{\partial z} = zB^{(1)} + \frac{2}{s_2^2}\frac{\partial A^{(1)}}{\partial z}. \tag{A32}$$

Adding the two expressions of (A32) and using (A29a) and (A27), one yields

$$\nabla_2^2(A^{(0)} + zA^{(1)}) = B^{(0)} + zB^{(1)} = B = \nabla_2^2 A. \tag{A33}$$

Thus,

$$\nabla_2^2(A - A^{(0)} - zA^{(1)}) = 0. \tag{A34}$$

Setting

$$A^{(2)} = A - A^{(0)} - zA^{(1)}. \tag{A35}$$

From (A34) and (A35), we get

$$\nabla_2^2 A^{(2)} = 0. \tag{A36}$$

Equations (A35), (A31b), (A30b) and (A36) together are exactly (A26). The lemma is proved.